

Note

Short Proofs on the Matching Polyhedron

A. SCHRIJVER

*Instituut voor Actuarie en Econometrie, Universiteit van Amsterdam,
Jodenbreestraat 23, 1011 NH Amsterdam, Holland*

Communicated by the Editors

Received September 14, 1981

A short proof of Edmonds' matching polyhedron theorem and the total dual integrality of the associated system of linear inequalities, proved first by W. H. Cunningham and A. B. Marsh (*Math. Programming Stud.* 8 (1978), 50–72), is given.

1. THE MATCHING POLYHEDRON

Let $G = (V, E)$ be an undirected graph, with $|V|$ even, and let P be the associated *perfect matching polytope*, i.e., P is the convex hull of the incidence vectors (in $\{0, 1\}^E$) of perfect matchings in G . In this paper we give a short proof of Edmonds' matching polyhedron theorem [3], which states that P is equal to the set of vectors x in \mathbb{R}^E satisfying

- (i) $x(e) \geq 0 \quad (e \in E),$ (1)
- (ii) $x(\delta(v)) = 1 \quad (v \in V),$
- (iii) $x(\delta(V')) \geq 1 \quad (V' \subseteq V, |V'| \text{ odd}).$

(Here $\delta(V')$ is the set of edges of G intersecting V' in exactly one point, $\delta(v) := \delta(\{v\})$, and $x(E') := \sum_{e \in E'} x(e)$ for $E' \subseteq E$.)

Let P' be the set of vectors in \mathbb{R}^E satisfying (1). As the incidence vector of any perfect matching satisfies (1), it follows that $P \subseteq P'$ —the content of Edmonds' theorem is the converse inclusion; equivalently, that the polytope defined by (1) has integer vertices only. (For other proofs, see Lovász [5] and Seymour [8]. For applications, see Naddef and Pulleyblank [6].)

EDMONDS' MATCHING POLYHEDRON THEOREM. *The perfect matching polytope is determined by the inequalities (1).*

Proof. Let G be a smallest graph with $P' \not\subseteq P$ (that is, $|V| + |E|$ is minimal), and let x be a vertex of P' not contained in P . Then $0 < x(e) < 1$ for all $e \in E$ —otherwise, we could delete e from G to obtain a smaller counterexample. Moreover, $|E| > |V|$ —otherwise, either G is disconnected (in which case one of the components of G will be a smaller counterexample), or G has a point v of degree one (in which case the edge e incident with v has $x(e) = 1$), or G is an even circuit (for which the theorem trivially holds).

Since x is a vertex, there are $|E|$ independent constraints among (1) satisfied by x with equality, and hence there is a $V' \subseteq V$ with $|V'|$ odd, $|V'| \geq 3$, $|V \setminus V'| \geq 3$, and $x(\delta(V')) = 1$. Let G_1 and G_2 arise from G by contracting V' and $V \setminus V'$, respectively, and let x_1 and x_2 be the corresponding projections of x onto the edge sets of G_1 and G_2 , respectively. Since x_1 and x_2 satisfy inequalities (1) for the smaller graphs G_1 and G_2 , respectively, it follows that x_1 and x_2 can be decomposed as convex combinations of perfect matchings in G_1 and G_2 , respectively. These decompositions can be easily glued together to form a decomposition of x as a convex combination of perfect matchings, contradicting our assumption. (This glueing can be done, e.g., as follows: By the rationality of x (as it is a vertex of P'), there exists a natural number K such that, for $i = 1, 2$, Kx_i is the sum of the incidence vectors of the perfect matchings F_1^i, \dots, F_K^i of G_i (possibly with repetitions). Since, for each $e \in \delta(V')$, e is contained in $Kx(e)$ of the F_j^1 as well as in $Kx(e)$ of the F_j^2 , we may assume that $F_j^1 \cap F_j^2 \neq \emptyset$, for $j = 1, \dots, K$. It follows that Kx is the sum of the incidence vectors of the perfect matchings $F_1^1 \cup F_1^2, \dots, F_K^1 \cup F_K^2$ of G , and hence that x itself is a convex combination of perfect matchings in G .) ■

By a standard construction we now derive Edmonds' characterization of the *matching polytope*, i.e., of the convex hull of (not-necessarily perfect) matchings. Again, $G = (V, E)$ is an undirected graph, but now $|V|$ may be odd. Edmonds showed that the matching polytope is determined by the inequalities

$$\begin{aligned} \text{(i)} \quad & x(e) \geq 0 && (e \in E), \\ \text{(ii)} \quad & x(\delta(v)) \leq 1 && (v \in V), \\ \text{(iii)} \quad & x(\langle V' \rangle) \leq \frac{1}{2}(|V'| - 1) && (V' \subseteq V, |V'| \text{ odd}). \end{aligned} \tag{2}$$

(Here $\langle V' \rangle$ denotes the set of edges contained in V' .) Again it is clear that each vector in the matching polytope satisfies (2), as the incidence vector of each matching satisfies (2).

COROLLARY. *The matching polytope is determined by (2).*

Proof. Let $x \in \mathbb{R}^E$ satisfy (2). Let $G = (V^*, E^*)$ be a disjoint copy of G , where the copy of vertex v will be denoted by v^* , and the copy of edge e

($= \{v, w\}$) will be denoted by e^* ($= \{v^*, w^*\}$). Let \tilde{G} be the graph with vertex set $V \cup V^*$ and with edge set $E \cup E^* \cup \{\{v, v^*\} \mid v \in V\}$. Define $\tilde{x}(e) = \tilde{x}(e^*) = x(e)$ for e in E , and $\tilde{x}(\{v, v^*\}) := 1 - x(\delta(v))$, for v in V . Now condition (1) is easily derived for \tilde{x} with respect to \tilde{G} . (i) and (ii) are trivial. To prove (iii) we have to show, for $V_1, V_2 \subseteq V$ with $|V_1| + |V_2|$ odd, that $\tilde{x}(\delta(V_1 \cup V_2^*)) \geq 1$. Indeed, we may assume, without loss of generality, that $|V_1 \setminus V_2|$ is odd. Hence

$$\begin{aligned} \tilde{x}(\delta(V_1 \cup V_2^*)) &= \tilde{x}(\delta(V_1 \setminus V_2)) + \tilde{x}(\delta(V_2^* \setminus V_1^*)) \geq \tilde{x}(\delta(V_1 \setminus V_2)) \\ &= |V_1 \setminus V_2| - 2x(\langle V_1 \setminus V_2 \rangle) \geq 1, \end{aligned}$$

by (2)(iii).

Hence \tilde{x} is a convex combination of perfect matchings of \tilde{G} . By restriction to x and G it follows that x is a convex combination of matchings in G . ■

2. DUAL INTEGRALITY

The preceding corollary is equivalent to: the polytope defined by (2) has integer vertices only. In other words, for each “weight” function $w \in \mathbb{R}^E$, the linear program

$$\begin{aligned} \max w^\top x, \\ \text{subject to (2)} \end{aligned} \tag{3}$$

has an integer optimal solution. The dual program is

$$\begin{aligned} \min \sum_{v \in V} y(v) + \sum_{V' \in \mathcal{O}} z(V') \frac{1}{2}(|V'| - 1) \\ \text{subject to} \\ y(v) \geq 0 \quad (v \in V), \\ z(V') \geq 0 \quad (V' \in \mathcal{O}), \\ \sum_{v \in e} y(v) + \sum_{\substack{V' \in \mathcal{O} \\ e \subseteq V'}} z(V') \geq w(e) \quad (e \in E), \end{aligned} \tag{4}$$

where \mathcal{O} denotes the collection of all subsets of V of odd size. Cunningham and Marsh [2] (cf. Schrijver and Seymour [7]) showed that if w is integral, this dual program also has an integer optimal solution, that is, the system of inequalities (2) is *totally dual integral* (cf. Edmonds and Giles [4]). Note that if we take $w \equiv 1$, this implies the Tutte–Berge theorem (Tutte [9], Berge

[1]): the maximum size of a matching in G is equal to the minimum value of $|V| - \frac{1}{2}(|V'| + \sigma(V'))$, where V' ranges over the subsets of V , and where $\sigma(V')$ denotes the number of odd-sized components of the subgraph induced by V' .

THEOREM. *If w is integral, then problem (4) has an integer optimal solution.*

Proof. We may assume that w is nonnegative. Suppose that G and $w \in \mathbb{Z}_+^E$ form a smallest counterexample, i.e., that (4) has no integer optimal solution and that $|V| + |E| + \sum_{e \in E} w(e)$ is as small as possible. Then $w(e) \geq 1$ for all e in E , otherwise, we could delete e . Let \mathcal{F} be the collection of those matchings in G whose incidence vector achieves the maximum (3). Then for each vertex v there is a matching F in \mathcal{F} not covering v . Otherwise, we could decrease the weights of the edges incident with v by one, thus decreasing the maximum (3), and therefore also the minimum (4), by one. For this smaller weight function there is an integer optimal solution y, z for (4). By increasing $y(v)$ by one we obtain an integer optimal solution for w .

Now let y, z be an optimal solution for (4) with $\sum_{V' \in \mathcal{C}} z(V')|V'| \cdot |V \setminus V'|$ as small as possible.

First, $y \equiv 0$, since if $y(v) > 0$, by complementary slackness each F in \mathcal{F} covers v .

Secondly, if $V', V'' \in \mathcal{C}$ with $z(V') > 0$, $z(V'') > 0$, and $V' \cap V'' \neq \emptyset$, then $V' \subseteq V''$ or $V'' \subseteq V'$. For let $v \in V' \cap V''$, and take F in \mathcal{F} not covering v . Then, by complementary slackness, $\frac{1}{2}(|V'| - 1)$ edges of F are contained in V' , and $\frac{1}{2}(|V''| - 1)$ edges of F are contained in V'' . This directly implies that $|V' \cap V''|$ and $|V' \cup V''|$ are odd. Suppose now that $V' \setminus V'' \neq \emptyset \neq V'' \setminus V'$. Let $\varepsilon = \min\{z(V'), z(V'')\}$, and redefine z by

$$\begin{aligned} z(V') &:= z(V') - \varepsilon, & z(V'') &:= z(V'') - \varepsilon, \\ z(V' \cap V'') &:= z(V' \cap V'') + \varepsilon, & z(V' \cup V'') &:= z(V' \cup V'') + \varepsilon, \end{aligned} \quad (5)$$

and let z be unchanged in the other components. One easily checks that the new y, z again is an optimal solution for (4), and moreover that $\sum_{V' \in \mathcal{C}} z(V')|V'| \cdot |V \setminus V'|$ is smaller than before, contradicting our assumption.

Finally, z is integral, for suppose $z(V')$ is not an integer, with $V' \in \mathcal{C}$ and $|V'|$ as large as possible. Let V_1, \dots, V_k be the maximal elements (with respect to inclusion) of $\{V'' \in \mathcal{C} \mid z(V'') > 0, V'' \subset V'\}$. So V_1, \dots, V_k are pairwise disjoint. Now let r be the fractional part of $z(V')$, and reset

$$z(V') := z(V') - r, \quad z(V_i) := z(V_i) + r \quad (\text{for } i = 1, \dots, k). \quad (6)$$

One easily checks that y, z again is a feasible solution for (4) (using that w is integral), attaining a smaller criterion value, which contradicts that the original y, z is optimal. ■

We leave it to the reader to derive from this theorem that the dual of the linear program: $\max w^T x$ subject to (1), has a half-integer optimal solution for each w in \mathbb{Z}^E (the graph K_4 shows that there do not always exist integer optimal solutions).

ACKNOWLEDGMENT

I thank the referee for helpful comments.

Note added in proof. Bill Cunningham (Bonn) informed the author that Jack Edmonds found a similar proof for the matching polyhedron theorem. An alternative short proof of both the matching polyhedron theorem and the dual integrality is given in: A. Schrijver, Min-max results in combinatorial optimization, in "Mathematical Programming Bonn 1982: The State of the Art," Springer-Verlag, Heidelberg, 1983.

REFERENCES

1. C. BERGE, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris Ser. A-B* **247** (1958), 258–259.
2. W. H. CUNNINGHAM AND A. B. MARSH III, A primal algorithm for optimal matching, *Math. Programming Stud.* **8** (1978), 50–72.
3. J. EDMONDS, Maximum matching and a polyhedron with 0, 1 vertices, *J. Nat. Bur. Stand. Sect. B* **69** (1965), 125–130.
4. J. EDMONDS AND R. GILES, A min-max relation for submodular functions on graphs, *Ann. Discrete Math.* **1** (1977), 185–204.
5. L. LOVÁSZ, Graph theory and integer programming, *Ann. Discrete Math.* **4** (1979), 141–158.
6. D. NADDEF AND W. R. PULLEYBLANK, Matchings in regular graphs, *Discrete Math.* **34** (1981), 283–291.
7. A. SCHRIJVER AND P. D. SEYMOUR, "A Proof of Total Dual Integrality of Matching Polyhedra," Math. Centre Report, ZN 79, Amsterdam, 1977.
8. P. D. SEYMOUR, On multi-colourings of cubic graphs, and conjectures of Fulkerson and Tutte, *Proc. London Math. Soc. (3)* **38** (1979), 423–460.
9. W. T. TUTTE, The factorization of linear graphs, *J. London Math. Soc. (3)* **22** (1947), 107–111.